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RANDOM VIBRATION OF DAMPED MODIFIED BEAM SYSTEMS

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In the work presented here, a method is developed to predict the stationary random response of a beam which has been modified by the attachment of a damped, lumped assembly of linear mechanical elements. The initial development treats a general beam system with attached linear elements. Two examples are presented with a cantilever beam modified, respectively, by a tip damper and a damped vibration absorber attached at the tip. The attached vibration absorber presents an interesting optimization problem to find the damping that minimizes the mean-square motion at the beam tip.

1. INTRODUCTION

In this work, the random response of two damped, modified beam systems is considered namely a cantilever beam with a viscous damper attached at the free end and that of a damped dynamic vibration absorber attached at the free end. There is a wealth of literature having to do with damped modified systems. The first report of such activity was by Duncan [1] in which he defined the concept of receptance and used receptances to calculate natural frequencies of simple combined mechanical systems. A nice explanation of this technique is given in the book of Bishop and Johnson [2]. Early papers by Young considered the combination of distributed and lumped parameter mechanical elements [3, 4]. There have been many papers over the past three decades which have considered the dynamics of vibratory systems in the presence of modifications involving lumped parameter elements. The pivotal works are those of Wissenberger [5], Jacquot and Soedel [6], Pomazal and Snyder [7], Dowell [8], and Hallquist and Snyder [9].

The effect of elastic constraints on the random vibration of damped linear structures was considered by Howell [10]. More recently, the random vibration of combined linear systems has been considered by Bergman and Nicholson [11] in which they employ the normal mode method and Green's functions to express the cross-correlation functions and cross-spectral density functions of the beam response. Still more recently, Kareem and Sun [12] have considered the problem of random vibration of a structure carrying a tank of sloshing fluid which is modelled as a series of parallel attached linear oscillators. The theory given adds an additional number of degrees of freedom to handle the additional attached oscillators. The solution to the resulting problem is then given as the usual lumped parameter eigensolution and does not build on knowledge of the problem prior to the addition of the tank of fluid. The random vibration of a damped tapered beam carrying masses is treated in the work of Yadav *et al.* wherein the first and second order statistics of the responses are calculated for a cantilever beam with a base excitation [13].

In this work, the author uses the known eigenstructure of an arbitrary beam to infer the response character of the system composed of the beam and an attached damped substructure. An alternative approach would be to incorporate additional degrees of freedom and solve a new eigenvalue–eigenvector problem to calculate the mean-square response.

2. TRANSFER FUNCTION FOR THE COMBINED SYSTEM

The methodology followed here is that of reference [14]. Consider a Bernoulli–Euler beam which is driven by two forcing functions one of which is uniform in space denoted by $w(t)$ and a concentrated force $p(t)$ generated by an assembly of attached linear passive elements located at $x = a$ as illustrated in Figure 1. The beam is a linear, time-invariant system and as such the concept of a transfer function and the principle of superposition are valid. The response at some point x on the beam is thus given by

$$Y(x, s) = G_1(x, s)P(s) + G_2(x, s)W(s), \tag{1}$$

where $P(s)$ and $W(s)$ are, respectively, the Laplace transforms of $p(t)$ and $w(t)$. If the force $P(s)$ is generated by a collection of passive, linear elements attached to the beam at location $x = a$,

$$P(s) = -H(s)Y(a, s), \tag{2}$$

where $H(s)$ may be thought of as the displacement driving-point impedance of the attached mechanical elements. Relation (2) may be substituted into equation (1) to yield

$$Y(x, s) = -G_1(x, s)H(s)Y(a, s) + G_2(x, s)W(s). \tag{3}$$

Since equation (3) holds for all x, then it must hold at $x = a$, so that

$$Y(a, s) = -G_1(a, s)H(s)Y(a, s) + G_2(a, s)W(s). \tag{4}$$

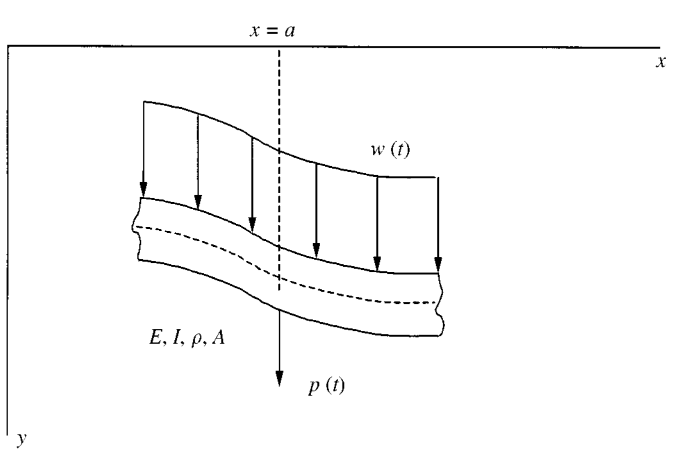

Figure 1. Section of the beam with two forcing functions.

This relation may be solved for the ratio of $Y(a,s)/W(s)$ to yield the transfer function

$$M(a,s) = \frac{Y(a,s)}{W(s)} = \frac{G_2(a,s)}{1 + G_1(a,s)H(s)}. \tag{5}$$

In turn $Y(a,s)$ in equation (3) may be eliminated using relation (5) to give

$$Y(x,s) = G_2(x,s)W(s) - G_1(x,s)M(a,s)H(s)W(s), \tag{6}$$

so the transfer function between $W(s)$ and $Y(x,s)$ is

$$M(x,s) = \frac{Y(x,s)}{W(s)} = G_2(x,s) - G_1(x,s)M(a,s)H(s), \tag{7}$$

where $M(a,s)$ has already been given in relation (5).

3. THE STATIONARY RANDOM VIBRATION PROBLEM

If a damped modified beam is forced by a single forcing function $w(t)$ which is a zero mean, second order, stationary random process with a power spectral density $S_w(\omega)$, it is the goal here to predict the mean-square response at any point x on the beam. In the previous section, it has been shown that there exists a transfer function $M(x, s)$ relating the response at a point x, $Y(x,s)$, to forcing function $W(s)$. If the modification to the beam is of a passive, dissipative nature then $M(x,s)$ has only poles which lie in the left half of the complex s-plane and hence the frequency response function $M(x, \mathrm{j}\omega)$ has meaning. If the power spectrum of the input $S_w(\omega)$ and $M(x, \mathrm{j}\omega)$ are known then the power spectral density of the response at point $x, y(x,t)$ is

$$S_y(x,\omega) = |M(x, \mathrm{j}\omega)|^2 S_w(\omega). \tag{8}$$

Note that this response power spectrum is a function of location x and hence will be different at different locations. The mean-square response at location x is given by integrating the power spectral density or

$$\sigma_y^2(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |M(x, \mathrm{j}\omega)|^2 S_w(\omega)\,\mathrm{d}\omega. \tag{9}$$

4. TRANSFER FUNCTIONS FOR THE BEAM

Consider the motion of the beam due to the concentrated force $p(t)$ located at $x = a$. The beam is of length L, mass per unit length ρA, with bending stiffness EI. The boundary conditions are any combination of pinned, clamped or free. The problem is described by the Bernoulli–Euler equation for small motions of slender beams,

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = p(t)\,\delta(x - a). \tag{10}$$

The spatial boundary conditions dictate the eigenfunctions $\phi_i(x)$ and the eigenvalues β_i which are related to the natural frequencies by

$$\omega_i^2 = \beta_i^4 \frac{EI}{\rho A}. \tag{11}$$

The spatial Dirac delta function can be expanded in a generalized Fourier series of the eigenfunctions as

$$\delta(x-a) = \sum_{i=1}^{\infty} c_i \phi_i(x), \tag{12}$$

where due to the orthogonality of the eigenfunctions the c_i are given by

$$c_i = \frac{\int_0^L \delta(x-a)\phi_i(x)\,\mathrm{d}x}{\int_0^L \phi_i^2(x)\,\mathrm{d}x} \tag{13}$$

and the shifting property of the Dirac delta function yields

$$\delta(x-a) = K_i \sum_{i=1}^{\infty} \phi_i(a)\phi_i(x), \tag{14}$$

where the K_i is the reciprocal of the denominator integral of equation (13) which is tabulated by Felgar [16] for various beam boundary conditions. The solution to equation (10) may also be expanded in a series in the eigenfunctions as

$$y(x,t) = \sum_{i=1}^{\infty} q_i(t)\phi_i(x). \tag{15}$$

Substitution of equations (15) and (12) into equation (10) and equating term by term gives a set of ordinary differential equations in the generalized co-ordinates as

$$\rho A \ddot{q}_i + EI\beta_i^4 q_i = c_i p(t), \quad i = 1, 2, \dots\ . \tag{16}$$

If the Laplace transform is taken and the initial conditions are ignored since only the sinusoidal steady state solution is desired, the result is

$$Q_i(s) = \frac{c_i P(s)}{\rho A s^2 + EI\beta_i^4}. \tag{17}$$

The transfer function between the forcing function $P(s)$ and the beam displacement at location x is

$$G_1(x,s) = \frac{Y(x,s)}{P(s)} = \sum_{i=1}^{\infty} \frac{c_i \phi_i(x)}{\rho A s^2 + EI\beta_i^4}. \tag{18}$$

In a similar fashion, the transfer function between the uniform force $W(s)$ and the displacement of the beam at location x can be shown to be

$$G_2(x,s) = \frac{Y(x,s)}{W(s)} = \sum_{i=1}^{\infty} \frac{d_i \phi_i(x)}{\rho A s^2 + EI\beta_i^4}, \tag{19}$$

where the d_i are defined by

$$d_i = \frac{\int_0^L \phi_i(x)\,\mathrm{d}x}{\int_0^L \phi_i^2(x)\,\mathrm{d}x}. \tag{20}$$

As mentioned previously, the numerator and denominator integrals have been tabulated [16] and are functions of the beam boundary conditions.

5. THE CANTILEVER BEAM

Consider now the developments of the previous section applied to a cantilever beam. The eigenfunctions for the beam are

$$\phi_i(x) = \cosh\beta_i x - \cos\beta_i x - \alpha_i(\sinh\beta_i x - \sin\beta_i x), \tag{21}$$

where the values of $\beta_i L$ are the ordered positive roots of the transcendental equation

$$1 + \cos\beta L \cos\beta L = 0. \tag{22}$$

The values of $\beta_i L$ are tabulated by Young and Felgar [15] as are the values of α_i for various beam boundary conditions. The integrals involved in the evaluation of the c_i and d_i (equations (13) and (20)) have been tabulated by Felgar [16] for all ordinary boundary conditions. The value of K_i in equation (14) for a cantilever beam is $1/L$ so the c_i are

$$c_i = \frac{\phi_i(a)}{L}. \tag{23}$$

In a similar fashion, the d_i from relation (17) are

$$d_i = \frac{2\alpha_i}{\beta_i L}. \tag{24}$$

The transfer function $G_1(x,s)$ is then

$$G_1(x,s) = \frac{1}{L}\sum_{i=1}^{\infty} \frac{\phi_i(a)\phi_i(x)}{\rho A s^2 + EI\beta_i^4} \tag{25}$$

and the transfer function $G_2(x,s)$ between the tip force and the motion at a point x is

$$G_2(x,s) = 2\sum_{i=1}^{\infty} \frac{\alpha_i \phi_i(x)}{\beta_i L(\rho A s^2 + EI\beta_i^4)}. \tag{26}$$

6. EXAMPLE 1 (CANTILEVER BEAM WITH A TIP DAMPER)

In this case, the cantilever beam is modified by a viscous damper at the free end as illustrated in Figure 2 so the transfer function for the attached element is

$$H(s) = bs \tag{27}$$

and $a = L$. The transfer function between $W(s)$ and the motion of the tip is given by relation (5) and the previously derived transfer functions of relations (25) and (26), to be

$$M(L, s) = \frac{Y(L, s)}{W(s)} = \frac{(2/L)\sum_{i=1}^{\infty} \alpha_i \phi_i(L)/\beta_i(\rho A s^2 + EI\beta_i^4)}{1 + (bs/L)\sum_{i=1}^{\infty} \phi_i^2(L)/(\rho A s^2 + EI\beta_i^4)}. \tag{28}$$

The non-dimensional frequency response function is

$$\frac{EIM(L,\ \mathrm{j}\omega)}{L^4} = \frac{2\sum_{i=1}^{\infty} \alpha_i \phi_i(L)/(\beta_i L)^5(1 - (\omega/\omega_i)^2)}{1 + (bL^3\mathrm{j}\omega/EI)\sum_{i=1}^{\infty} \phi_i^2(L)/(\beta_i L)^4(1 - (\omega/\omega_i)^2)}. \tag{29}$$

Let us now non-dimensionalize frequency by defining the frequency ratio

$$f = \omega/\omega_1 \tag{30}$$

and γ_i to be the ratio of the ith natural frequency to the first natural frequency

$$\gamma_i = \omega_i/\omega_1. \tag{31}$$

Also define a non-dimensional damping coefficient B to be

$$B = \frac{bL(\beta_1 L)^2}{\sqrt{EI\rho A}}. \tag{32}$$

Using equations (30)–(32) the dimensionless tip-frequency response function in terms of the dimensionless frequency variable f is

$$\frac{EIM(L,\ \mathrm{j}f)}{L^4} = \frac{\sum_{i=1}^{\infty} 2\alpha_i \phi_i(L)/(\beta_i L)^5(1 - (f/\gamma_i)^2)}{1 + \mathrm{j}fB\sum_{i=1}^{\infty} \phi_i^2(L)/(\beta_i L)^4(1 - (f/\gamma_i)^2)}. \tag{33}$$

With this being known, the frequency response of any other point on the beam can be evaluated using relation (7).

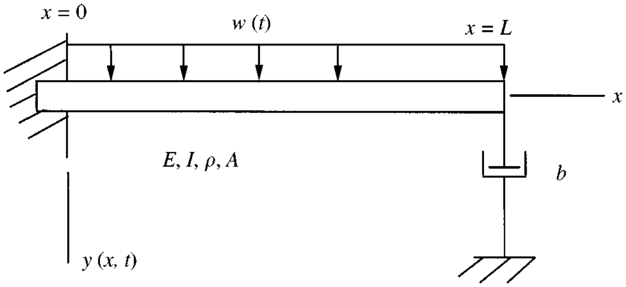

Figure 2. Cantilever beam with a tip damper.

Assume now that the beam is driven by white noise $w(t)$ with constant power spectral density S_w. Thus, the dimensionless tip-response power spectral density is, from relation (8),

$$\frac{(EI)^2 S_y(L,\omega)}{L^8 S_w} = \left| \frac{\sum_{i=1}^{\infty} 2\alpha_i \phi_i(L)/(\beta_i L)^5(1-(f/\gamma_i)^2)}{1 + \mathrm{j} f B \sum_{i=1}^{\infty} \phi_i^2(L)/(\beta_i L)^4(1-(f/\gamma_i)^2)} \right|^2. \tag{34}$$

This function is illustrated in Figure 3 and it is interesting to note that the effect of increasing the damping parameter B is to lower the peaks in the tip-response power spectral density function. Employing relation (7) the dimensionless spectral density of the midpoint of the beam is as illustrated in Figure 4. Note that the effect of increasing the damping is to eliminate the resonances near the cantilever natural frequencies and to induce new resonances at frequencies that approach those of a clamped–pinned beam.

One issue of importance is the convergence of the series in relation (33). It is seen that both series converge rapidly due to the denominator factor of $\beta_i L$ of powers four and five respectively. All calculations presented here were accomplished with five terms, however, they were also done with six terms with essentially no discernable difference in both the numerical and graphical results.

To evaluate the mean-square motions the integral of relation (9) must be evaluated. This means that the modified system poles must be calculated so the residue theorem can be employed. Although using MATLAB this task is reasonable, it is perhaps easier to evaluate the integral numerically using MATLAB by employing the trapezoidal rule or Simpson's rule. In this case, the trapezoidal rule is employed and the dimensionless mean-square motion defined as $[\sigma_y(L)EI]^2/S_w\omega_1 L^8$. The results of the numerical integrations are shown in Figure 5 as a function of the dimensionless damping parameter B.

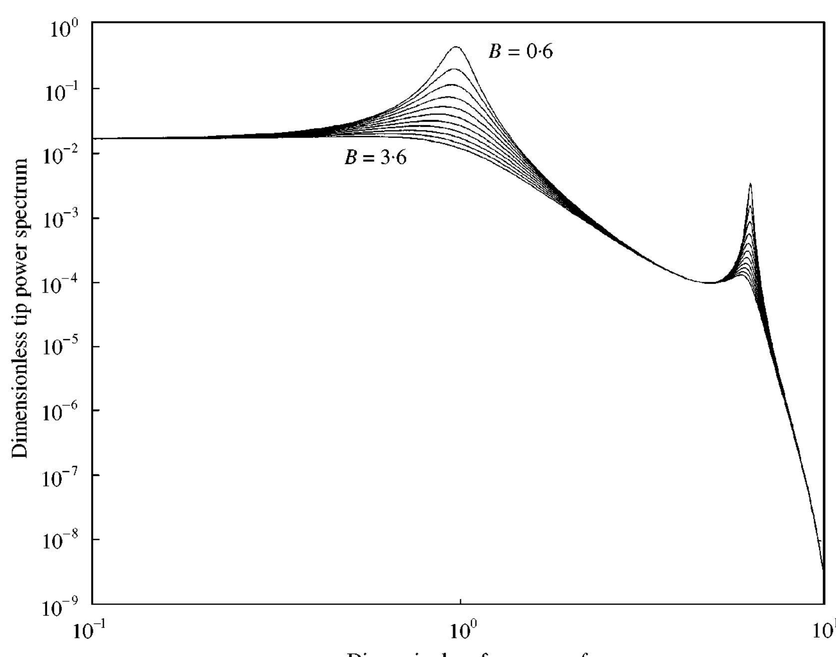

Figure 3. Dimensionless power spectral density of tip response for the cantilever beam with a tip damper from relation (34).

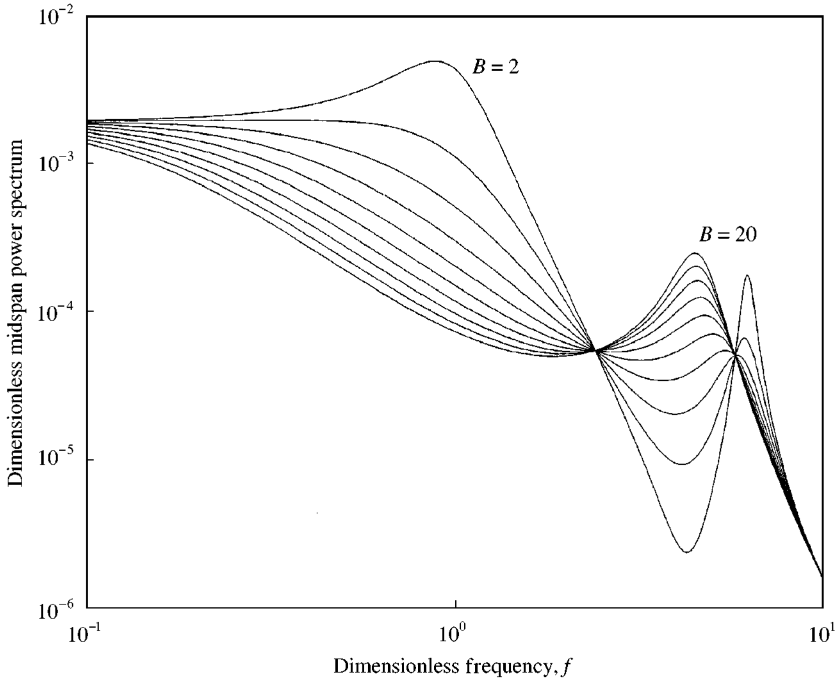

Figure 4. Dimensionless midspan displacement power spectral density for the cantilever beam with a tip damper from relation (7).

When performing the integration numerically, care must be taken to use a small enough dimensionless frequency increment to assure that the spectral energy under lightly damped resonance peaks is captured. This may be checked by halving the increment and repeating the integration process.

Similar results for the beam midpoint are shown in Figure 6. It is interesting to note that for larger values of B than illustrated the mean-square motion begins to increase. This occurs because for large B the tip motion approaches zero, the energy dissipation becomes small and thus the mean-square motion of the interior points of the beam increases. For large B the beam appears to be a clamped–pinned beam.

7. EXAMPLE 2 (CANTILEVER BEAM WITH A DYNAMIC VIBRATION ABSORBER)

In this example a cantilever beam is modified by a damped dynamic vibration absorber attached at the free end as illustrated in Figure 7. The transfer function relating force to displacement at the point of attachment is

$$H(s) = \frac{ms^2(bs + k)}{ms^2 + bs + k}. \tag{35}$$

The transfer function between $W(s)$ and the tip motion $Y(L, s)$ is given by expression (5) after substitution of relations (18) and (19) for $G_1(L, s)$ and $G_2(L, s)$ and relation (35) for $H(s)$. It is

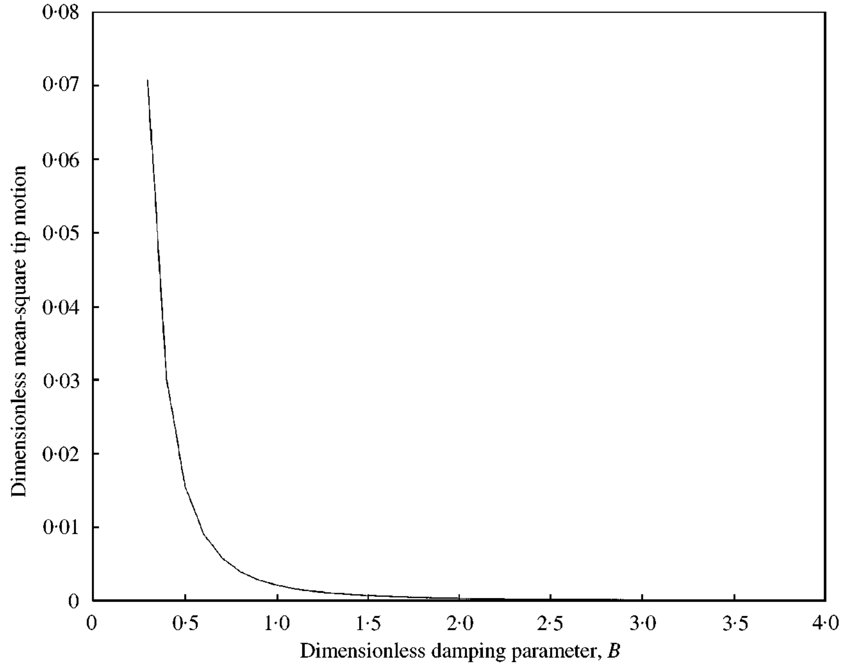

Figure 5. Dimensionless mean-square tip motion as a function of dimensionless damper coefficient given by integration of the power spectrum of Figure 3.

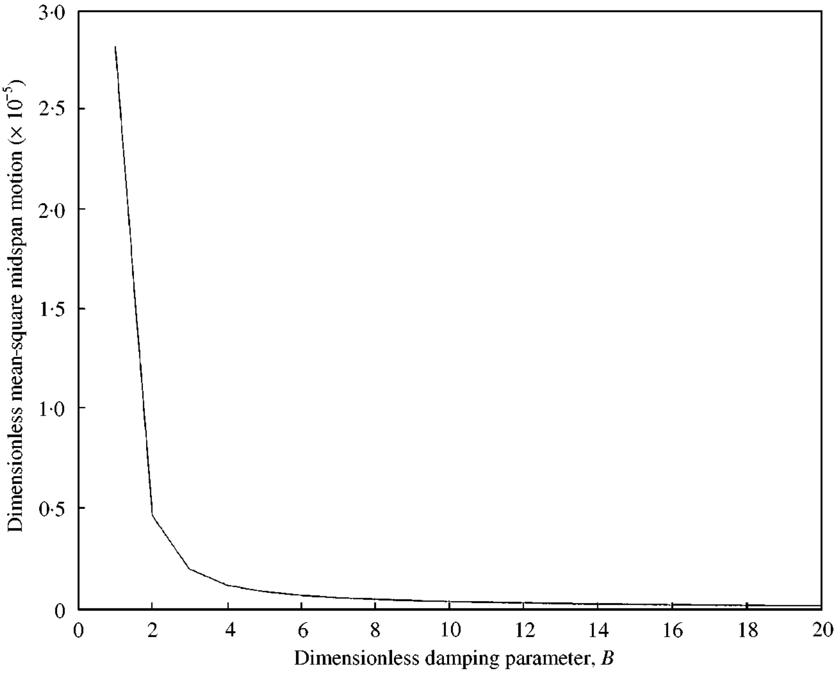

Figure 6. Dimensionless mean-square midspan motion as a function of dimensionless damper coefficient by integration of the power spectrum of Figure 4.

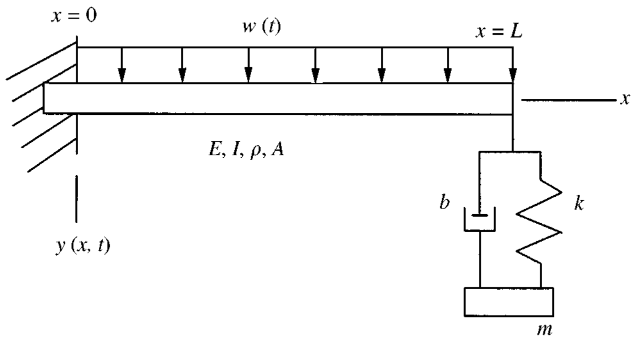

Figure 7. Cantilever beam with an attached dynamic vibration absorber at the tip.

given by

$$M(L,s)=\frac{Y(L,s)}{W(s)}=\frac{(2/L)\sum_{i=1}^{\infty}\alpha_i\phi_i(L)/\beta_i(\rho As^2+EI\beta_i^4)}{1+(ms^2(bs+k)/(ms^2+bs+k))(1/L)\sum_{i=1}^{\infty}\phi_i^2(L)/(\rho As^2+EI\beta_i^4)}. \tag{36}$$

The non-dimensional frequency response is then

$$\frac{EIM(L,\ \mathrm{j}\omega)}{L^4}=\frac{2\sum_{i=1}^{\infty}\alpha_i\phi_i(L)/(\beta_iL)^5(1-(\omega/\omega_i)^2)}{1-(m\omega^2(\mathrm{j}\omega b+k)/(-m\omega^2+\mathrm{j}\omega b+k))(L^3/EI)\sum_{i=1}^{\infty}\phi_i^2(L)/(\beta_iL)^4(1-(\omega/\omega_i)^2)}. \tag{37}$$

Assume that the dynamic absorber is tuned to the first natural frequency of the beam so that $k/m=\omega_a^2=\omega_1^2$ and, as in the previous example, the frequency variable can be scaled with respect to the first beam natural frequency so that

$$f=\frac{\omega}{\omega_1}=\omega\sqrt{\frac{m}{k}}. \tag{38}$$

The absorber damping ratio ζ can be defined as

$$\zeta=\frac{b}{2\sqrt{km}}. \tag{39}$$

It is common to define a mass ratio μ as the ratio of the absorber mass to the beam mass or

$$\mu=\frac{m}{\rho AL}. \tag{40}$$

The non-dimensional frequency response in terms of the parameters just defined is

$$\frac{EIM(L,\ \mathrm{j}f)}{L^4}=\frac{\sum_{i=1}^{\infty}2\alpha_i\phi_i(L)/(\beta_iL)^5(1-(f/\gamma_i)^2)}{1-\mu(f^2(\mathrm{j}2\zeta f+1)/(1-f^2+\mathrm{j}2\zeta f))\sum_{i=1}^{\infty}\phi_i^2(L)/\gamma_i^4(1-(f/\gamma_i)^2)}, \tag{41}$$

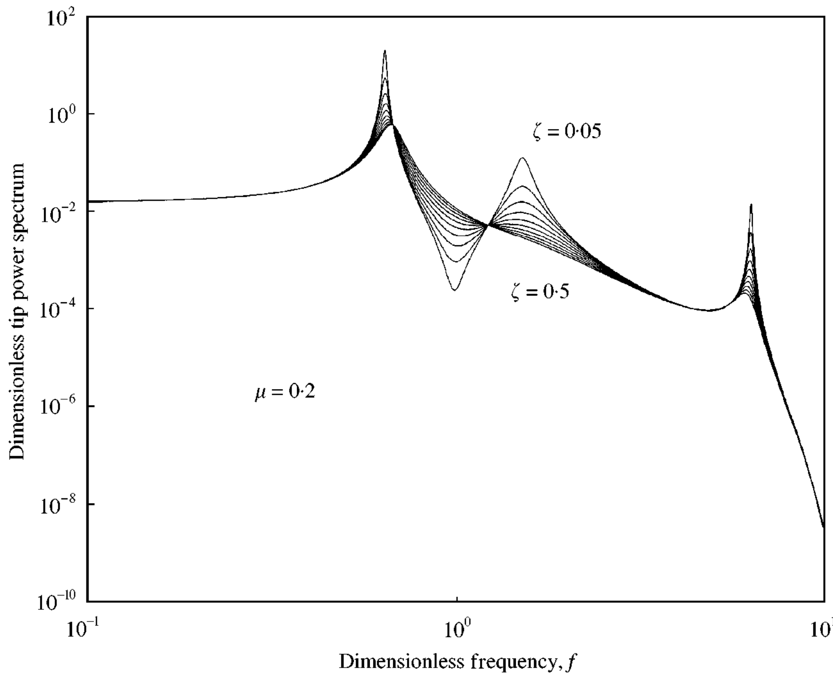

Figure 8. Dimensionless power spectral density of tip response for the cantilever beam with an attached dynamic vibration absorber at the tip from relation (42).

where $\gamma_i = \beta_i/\beta_1$ as in the previous example. If it is assumed that the forcing function $w(t)$ is white noise with constant power spectral density S_w then the dimensionless power spectral density function for the beam tip response is

$$\frac{(EI)^2 S_y(L, \mathrm{j}f)}{L^8 S_w} = \left| \frac{\sum_{i=1}^{\infty} 2\alpha_i \phi_i(L)/(\beta_i L)^5 (1 - (f/\gamma_i)^2)}{1 - \mu(f^2(\mathrm{j}2\zeta f + 1)/(1 - f^2 + \mathrm{j}2\zeta f)) \sum_{i=1}^{\infty} \phi_i^2(L)/\gamma_i^4 (1 - (f/\gamma_i)^2)} \right|. \quad (42)$$

This function for $\mu = 0{\cdot}2$ is shown in Figure 8 for various values of the absorber damping ratio ζ. As is typical, the function of the absorber tuned to the first beam natural frequency is to split the first resonance into two resonances while attenuating the response at the previous resonance at the first natural frequency of the beam. Figure 9 illustrates the result of relation (7) to calculate the power spectrum of the beam midpoint which has a character similar to that of the tip.

The dimensionless mean-square response of the beam tip which is $[\sigma_y(L)EI]^2/S_w\omega_1 L^8$ can then be given by relation (9) and has been evaluated for various damping ratios and mass ratios as illustrated in Figure 10. It is interesting to note that for a given mass ratio there is a damping ratio that yields a minimal mean-square motion and hence an interesting optimization problem presents itself. Examination of Figure 10 indicates that as the mass ratio μ is increased the value of the damping ratio ζ for a minimum mean-square tip motion increases. The damping ratio which yields the minimal mean-square tip motion is shown in Figure 11 as a function of the mass ratio μ.

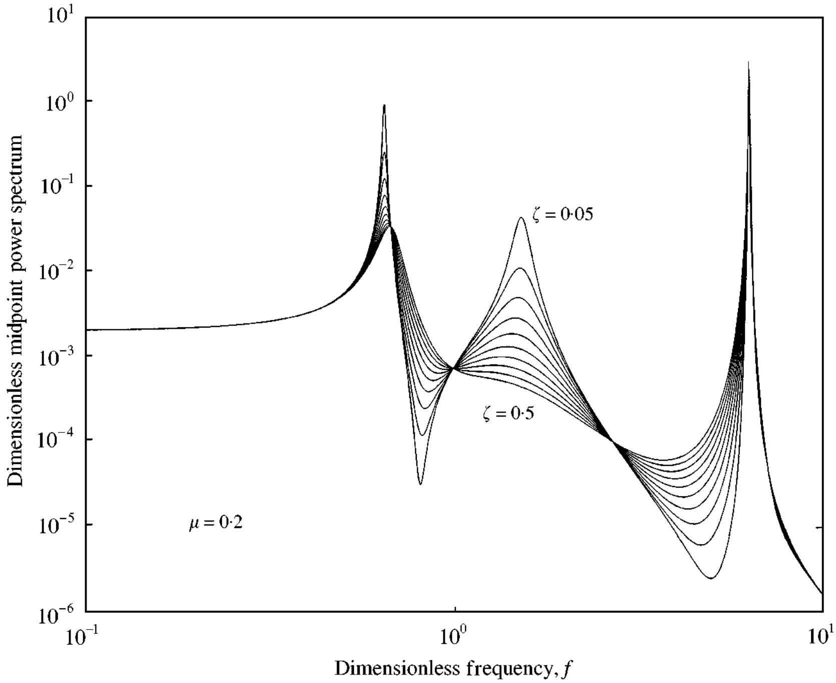

Figure 9. Dimensionless midspan displacement power spectral density for the cantilever beam with an attached dynamic vibration absorber at the tip from relation (7).

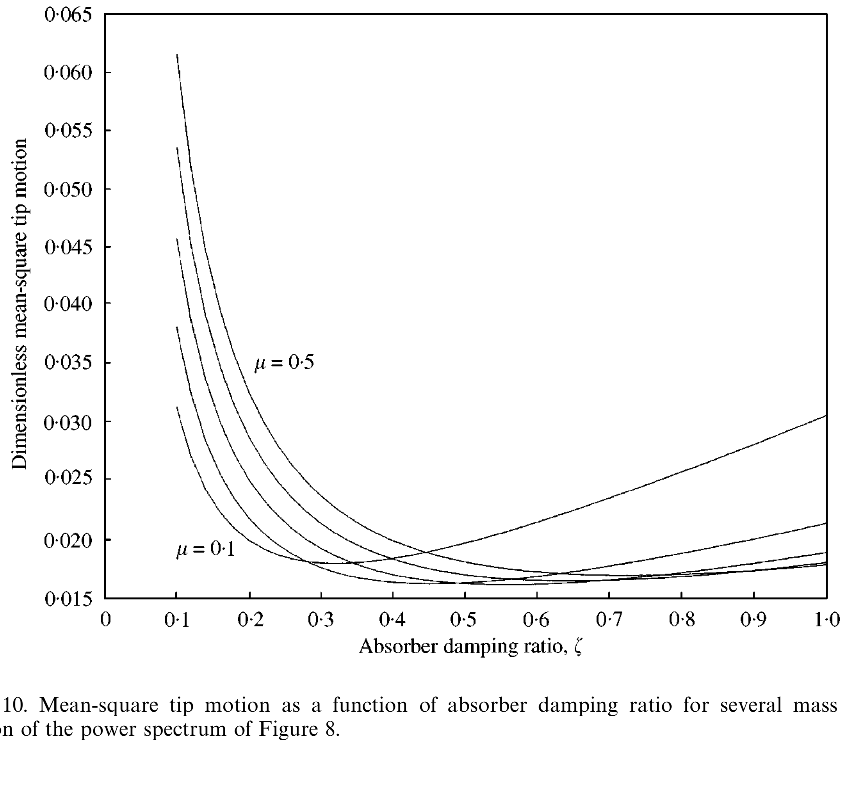

Figure 10. Mean-square tip motion as a function of absorber damping ratio for several mass ratios by integration of the power spectrum of Figure 8.

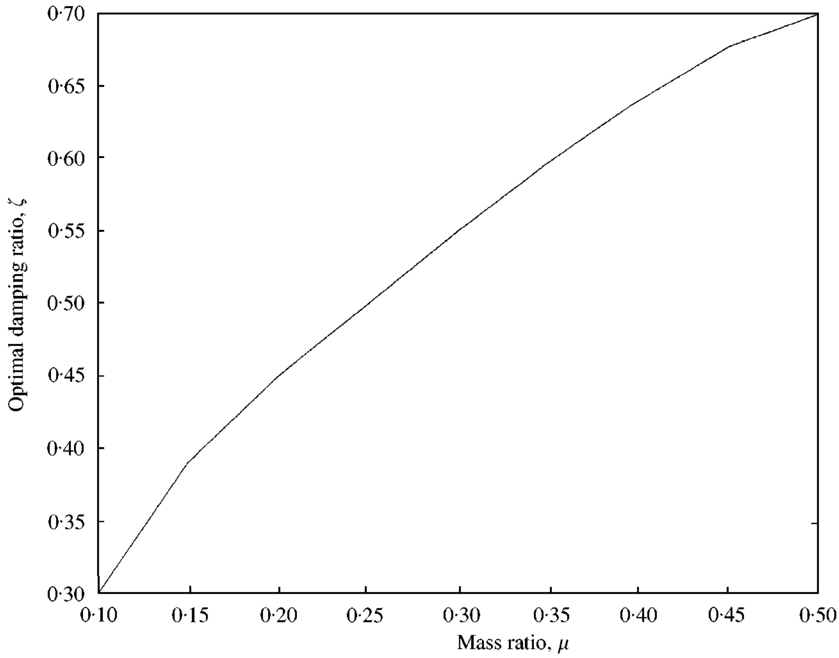

Figure 11. Optimal damping ratio for the absorber as a function of absorber mass ratio with the absorber tuned to the first beam natural frequency.

8. CONCLUSION

In this work, a method is developed to predict the response power spectral density function and mean-square response of a damped modified beam structure driven by a second order stationary random process. It is assumed that there is only a single modification to the system that is composed of an assembly of linear, lumped mechanical elements. This is not a serious limitation as the theory lends itself to multiple modifications. The procedure lends itself to the optimization of such attached assemblies so as to minimize the mean-square response at some location on the structure.

The method developed, as opposed to the particular examples presented, should be useful to practitioners that are attempting to predict the mean-square responses of complex structures to random force fields.

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